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LETTER TO THE EDITOR

Fricke–Klein geometry for the group $Sl(2, C)$

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Abstract. The homomorphisms from the free group F_2 with two generators to the complex unimodular group $Sl(2, C)$ are classified. The action induced by the automorphism group Φ_2 of F_2 on pairs of elements from $Sl(2, C)$ is expressed in a geometry inspired by Fricke and Klein.

The quantum mechanics of spin waves, electrons and phonons in non-periodic potentials involves a homomorphism h_1 from the free group F_2 with 2 generators to the groups $SU(2)$ and $SU(1, 1)$, and an induced action of the automorphism group Φ_2 of F_2 on the images under h_1. We refer to [7] for the groups F_2, Φ_2 and their generators. Some applications of this homomorphism are treated by Sutherland [8] and by Iguchi [4]. For the present geometric and algebraic approach we refer to [1, 5, 6], where reference is given to published work in this field. Since the group $Sl(2, C)$ contains both groups as subgroups, this group allows for a unified approach to homomorphisms $h_1 : F_2 \to Sl(2, C)$ and the induced action of Φ_2, which are presented in what follows.

For the groups $SU(2)$, $SU(1, 1)$, the Fricke–Klein geometry [5, 6] works as follows: Assign to three group elements g_1, g_2, g_3 from the groups $SU(2)$, $SU(1, 1)$ obeying $g_1g_2g_3 = e$ three dual (real) unit vectors ξ^1, ξ^2, ξ^3 whose scalar products are the traces of the group elements divided by two. To the generators P, σ, U [7] of Φ_2 associate a transformation of these three vectors: The images of ξ^1, ξ^2, ξ^3 under these generators are linearly expressed by these vectors and by their scalar products as

$$(\zeta^1\zeta^2\zeta^3) = (\xi^1\xi^2\xi^3)D(\rho) \tag{1}$$

with the transformation matrices

$$D(P) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 2\varepsilon(\xi^2 \cdot \xi^3) & 2\varepsilon(\xi^3 \cdot \xi^1) & 1 \end{bmatrix}$$
$$D(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2\varepsilon(\xi^2 \cdot \xi^3) & 1 \end{bmatrix} \tag{2}$$
$$D(U) = \begin{bmatrix} 2\varepsilon(\xi^3 \cdot \xi^1) & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The transformations are nonlinear due to the appearance of the scalar products. The number ε takes the value $\varepsilon = 1$ for $SU(2)$, and $\varepsilon = \pm 1$ for $SU(1, 1)$. In what follows we shall generalize these vectors and their transformations to the complex group $Sl(2, C)$.

We now consider the class types of $Sl(2, C)$. We begin with the remarks on the complex function $w = \cosh(z)$. The upper half-strip $z \in \langle 0, \infty) + i(0, \pi\rangle$ is mapped one-to-one into the upper half-plane, $w = \cosh(z) \in (-\infty, \infty) + i(0, \infty)$. We shall need, for the parametrization of group elements, only this half-strip and half-plane, and use this map for the inverse function. To the number z we assign a unique square root $z^{1/2}$ within the same half-strip. Next we study the eigenvalue problem of the matrix

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in Sl(2, C) \qquad \alpha\delta - \beta\gamma = 1. \tag{3}$$

The eigenvalues of g are

$$\lambda_{1,2} = \tfrac{1}{2}(\alpha + \delta) \pm [\tfrac{1}{4}(\alpha + \delta)^2 - 1]^{1/2} \qquad \lambda_1\lambda_2 = 1. \tag{4}$$

Clearly, the class types can be characterized by the traces of the group elements. Now we describe the *Lorentz-rotation class type* C_L: If $(\alpha + \delta) \neq \pm 2$, we can always choose λ_1 in the upper and λ_2 in the lower halfplane. With this choice of an orientation of the group parameters, and with the prescription given for the function $w = \cosh(z)$ and its inverse, we determine a unique complex number z: $\lambda_1 + \lambda_2 = 2\cosh(z)$. The real and imaginary part of z yield the exponential class-parameters of a commuting pair of an orthochronous Lorentz transformation and a rotation, respectively. The stability group of this representative diagonal form

$$g_L^0 = \begin{bmatrix} \exp(iz) & 0 \\ 0 & \exp(-iz) \end{bmatrix} \tag{5}$$

consists of *all* diagonal elements of $Sl(2, C)$. The coset generators c_L with respect to this stability group may be parametrized by two complex angles, which correspond to the Euler angles, are generated by the Pauli matrices σ_2 and σ_3 respectively, and yield the general element $g_L = c_L g_L^0 c_L^{-1}$ of the class. These two complex angles are the *in-class parameters.* Finally we note that the unique choice of $z^{1/2}$ in the half-strip described above determines a unique group element $g_L^{1/2}$.

If $\alpha + \delta = 2$ and $\beta = \gamma = 0$ we get $g = e$. Its class we denote by C_0. Next we consider the *Jordan class type* C_J characterized by $\alpha + \delta = 2$, $|\beta|^2 + |\gamma|^2 > 0$. Now the representative element is

$$g_J^0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

The stability group of g_J^0 consists of all upper triangular elements with diagonal entries 1, its cosets in $Sl(2, C)$ are given by the factors

$$c_J = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \tag{7}$$

of the Gauss decomposition. The parameters μ, a are the in-class parameters. By conjugation one finds the general element of the Jordan class type in the form

$$g_J = c_J g_J^0 c_J^{-1} = \begin{bmatrix} 1 - \mu a^2 & a^2 \\ -\mu^2 a^2 & 1 + \mu a^2 \end{bmatrix}. \tag{8}$$

A unique element $g_J^{1/2}$ is given by the replacement

$$g_J^0 \to (g_J^0)^{1/2} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \tag{9}$$

and conjugation with c_J. The element $-e \in Sl(2, C)$ forms a class $-C_0$ by itself. By multiplication it determines a second Jordan class type $-C_J$ with trace $\alpha + \delta = -2$ and otherwise similar class characterization.

The adjoint action and the group $SO(3, C)$ are considered next. We use a complex exponential parametrization of $Sl(2, C)$ with the Pauli matrices as a basis of the complex Lie algebra. On C^3 we use the scalar product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} a_i b_i. \tag{10}$$

This scalar product corresponds to the complex Killing metric of $sl(2, C)$. Then for all elements with the class type C_L, C_J we get

$$\begin{aligned} g &= \exp\left(-z \sum_{j=1}^{3} \eta_j \sigma_j\right) \\ &= \zeta e - \rho \sum_{j=1}^{3} \eta_j \sigma_j. \end{aligned} \tag{1}$$

Here $\boldsymbol{\eta}$ is a complex vector. For the class types C_L, C_J the numbers ζ, ρ are of the form

$$\begin{aligned} &\text{type } C_L\text{: } \zeta = \cosh z \qquad \rho = \sinh z \qquad \boldsymbol{\eta} \cdot \boldsymbol{\eta} = 1 \\ &\text{type } C_J\text{: } \zeta = 1 \qquad \boldsymbol{\eta} \cdot \boldsymbol{\eta} = 0. \end{aligned} \tag{12}$$

The complex adjoint action of $Sl(2, C)$ is defined by the conjugation $g' \to \mathrm{Ad}_g(g') = gg'g^{-1}$, it transforms the components of a complex vector $\boldsymbol{\eta}$ as

$$\eta_j \to \tilde{\eta}_j = \sum_{l=1}^{3} (\mathrm{Ad}_g)_{jl} \eta_l. \tag{13}$$

In terms of the four matrix elements $\alpha, \beta, \gamma, \delta$ of g, this complex matrix reads

$$(\mathrm{Ad}_g) = \begin{bmatrix} \frac{1}{2}(\alpha^2 - \beta^2 - \gamma^2 + \delta^2) & i(-\alpha^2 - \beta^2 + \gamma^2 + \delta^2)/2 & (-\alpha\beta + \gamma\delta) \\ i(\alpha^2 - \beta^2 + \gamma^2 - \delta^2)/2 & \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) & -i(\alpha\beta + \gamma\delta) \\ (-\alpha\gamma + \beta\delta) & i(\alpha\gamma + \beta\delta) & (\alpha\delta + \beta\gamma) \end{bmatrix}. \tag{14}$$

This matrix must conserve the complex scalar product equation (10) and hence must belong to $SO(3, C)$. The homomorphism from $Sl(2, C)$ to $SO(3, C)$ is two-to-one since $\mathrm{Ad}_{-g'} = \mathrm{Ad}_{g'}$. The homomorphism may also be seen as the symmetrized Kronecker square of the defining representation of $Sl(2, C)$, corresponding to the Young diagram {2}.

We are now ready to classify pairs of group elements from $Sl(2, C)$ modulo the adjoint action. Let g_1, g_2 denote two elements of $Sl(2, C)$ in the parametrization equation (11). Their product we denote by $g_3^{-1} = g_1 g_2$ and obtain from the multiplication of the Pauli matrices

$$\begin{aligned} \rho_3 \boldsymbol{\eta}^3 &= -\rho_1 \zeta_2 \boldsymbol{\eta}^1 - \zeta_1 \rho_2 \boldsymbol{\eta}^2 + \rho_1 \rho_2 i(\boldsymbol{\eta}^1 \times \boldsymbol{\eta}^2) \\ \zeta_3 &= \zeta_1 \zeta_2 + \rho_1 \rho_2 (\boldsymbol{\eta}^1 \cdot \boldsymbol{\eta}^2). \end{aligned} \tag{15}$$

Here the sign $\times$ denotes the standard vector product extended to complex vectors in C^3.

Given three group elements g_1, g_2, g_3 which obey $g_1 g_2 g_3 = e$, we define the Fricke–Klein geometry [5], [3] of this triple in terms of three complex unit vectors $\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \boldsymbol{\xi}^3$

with the properties

$$j \neq k: \qquad \xi^j \cdot \eta^k = 0$$
$$\tfrac{1}{2}\,\mathrm{tr}(g_i) = (\varepsilon_{ijk})^2(\xi^j \cdot \xi^k). \tag{16}$$

We shall prove the existence of these vectors by an explicit construction. First of all we note that, under the adjoint action applied to all three elements g_i, the vectors ξ^j transform according to the complex coadjoint action, which may be identified with the adjoint action since both are complex orthogonal. Therefore it suffices to consider pairs g_1, g_2 from the various class types. Here we restrict the discussion to the class types C_L, C_J since all other cases are obtained by multiplication with $-e$.

We now classify pairs modulo the adjoint action and use for this the results given earlier in this letter. One of the two group elements may then be chosen as the representative of its class type. The second one may, after this choice, still be conjugated with elements from the stability group of the first one. In table 1 we give the representative pairs from the class types C_L, C_J, the vectors η^i, the traces of the group elements, and the dual vectors ξ^i. Their scalar products obey equation (16). A fixed triple of vectors ξ^1, ξ^2, ξ^3 admits an interpretation in terms of a complex Coxeter group [2], acting on C^3 via the Weyl reflections

$$r_i : x \to x - 2(x \cdot \xi^i)\xi^i. \tag{17}$$

Multiplication of pairs of Weyl reflections yields

$$r_2 r_1 = \mathrm{Ad}_{g_3} \qquad r_3 r_2 = \mathrm{Ad}_{g_1} \qquad r_1 r_3 = \mathrm{Ad}_{g_2}. \tag{18}$$

This means that *the subgroup of $SO(3, C)$ generated by the adjoint action of g_1, g_2, g_3 is the subgroup of this Coxeter group, generated by products of an even number of reflections.*

Table 1. Representative pairs of group elements from $Sl(2, C)$ and their data.

Pair 1:	$\eta^1 = (0, 0, -1)$
	$\eta^2 = (-\sinh\delta, 0, -\cosh\delta)$
	$\tfrac{1}{2}\mathrm{tr}(g_1) = \cosh\alpha$
	$\tfrac{1}{2}\mathrm{tr}(g_2) = \cosh\beta$
	$\tfrac{1}{2}\mathrm{tr}(g_3) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta\cosh\delta$
	$\xi^1 = (i\sinh\beta\cosh\delta, \cos\beta, -i\sinh\beta\sinh\delta)$
	$\xi^2 = (-i\sinh\alpha, \cosh\alpha, 0)$
	$\xi^3 = (0, 1, 0)$
Pair 2:	$\eta^1 = (0, 0, -1)$
	$\eta^2 = (0, -i\rho, \rho)$
	$\tfrac{1}{2}\mathrm{tr}(g_1) = \cosh\alpha$
	$\tfrac{1}{2}\mathrm{tr}(g_2) = 1$
	$\tfrac{1}{2}\mathrm{tr}(g_3) = \cosh\alpha - \rho\sinh\alpha$
	$\xi^1 = (1, i\rho, -\rho)$
	$\xi^2 = (\sinh\alpha, i\cosh\alpha, 0)$
	$\xi^3 = (1, 0, 0)$
Pair 3:	$\eta^1 = (-\rho_1, -i\rho_1, 0)$
	$\eta^2 = (0, -i\rho_2, \rho_2)$
	$\tfrac{1}{2}\mathrm{tr}(g_1) = 1$
	$\tfrac{1}{2}\mathrm{tr}(g_2) = 1$
	$\tfrac{1}{2}\mathrm{tr}(g_3) = 1 - \rho_1\rho_2$
	$\xi^1 = (1, i(1+\rho_2), -(1+\rho_2))$
	$\xi^2 = (1+\rho_1, i(1+\rho_1), -1)$
	$\xi^3 = (1, i, 1)$

Conversely, it is possible to obtain the triple ξ^i from the adjoint action: Consider the vector $b^3 = \eta^1 \times \eta^2$ and the square roots $g_1^{1/2}, g_2^{1/2}$, and construct the images

$$b^1 = \mathrm{Ad}_{(g_2^{-1/2})}(b^3) \qquad b^2 = \mathrm{Ad}_{(g_1^{1/2})}(b^3). \tag{19}$$

Assume that b^3 has non-zero length. If ξ^3 is the unit vector proportional to b^3, then ξ^1, ξ^2 are the images of this vector according to equation (19).

Proof. Explicit computation of the adjoint action for the representative pairs given in table 1.

As a corollary of this construction we have: *The three vectors ξ^i are on a single orbit under the adjoint action of $Sl(2, C)$.*

For the group $SU(2)$, the real triples of unit vectors η^i and ξ^i form a pair of dual triangles. In figure 1 we give the intuitive planar representation of the spherical triangle spanned by η^1, η^2, η^3, of the dual vectors ξ^i, seen in tangential planes to this triangle, and of the real rotation angles α_i. The relations given in equation (19) for the subgroups $SU(2)$, $SO(3, R)$ can be seen from this figure.

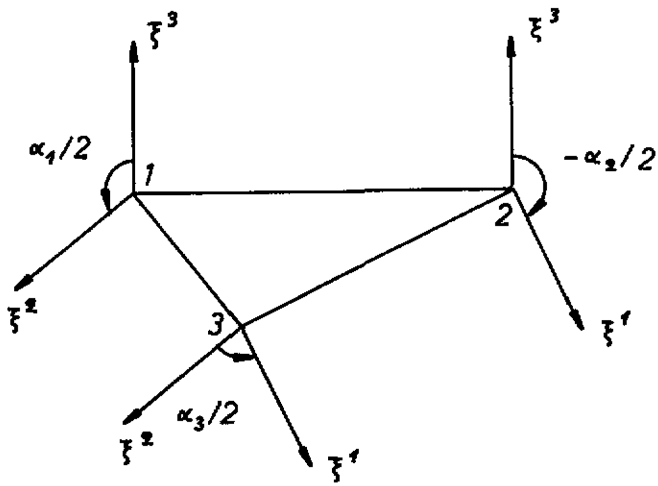

Figure 1. Planar representation of the spherical triangle formed by the three real vectors η^i, the three tangential vectors ξ^i, and the half-angles of rotation α_i for the Fricke–Klein geometry of $SU(2)$.

Consider now a homomorphism $h_1 : F_2 \to Sl(2, C)$. It suffices to study the homomorphisms h_1 modulo the adjoint action of $Sl(2, C)$. Then the homomorphisms h_1 are classified by the representative pairs given in table 1. In equations (1) and (2) we give a nonlinear representation of the automorphism group Φ_2 as an action on a triple of vectors ξ^1, ξ^2, ξ^3. We take $\varepsilon = 1$ and extend this nonlinear representation of Φ_2 to the group $Sl(2, C)$, by interpreting the vectors and matrices in C^3 rather than R^3. *These transformations provide an action of the group Φ_2 on pairs (and triples) of group elements from $Sl(2, C)$.* Since they represent the generators of Φ_2, this action is defined for the full automorphism group. The *trace-map* discussed in [1, 4, 8] results by computing from equation (2) the map of the three non-trivial scalar products $a_{ij} = (\xi^i \cdot \xi^j)$ under Φ_2.

The complex symmetric matrix $a=(a_{ij})$, whose entries are all the nine scalar products, play an important part for the Coxeter group. Written in terms of the traces by use of equation (16), which are called Fricke characters, its real counterpart was considered first by Fricke and Klein [3] and related to the group commutator. Since the generating matrices equation (2) all have determinant ± 1, the determinant of a is *an invariant under all trace maps.* We refer to [5, 6] for algebraic and geometric relations due to Nielsen, to Fricke and Klein and to other authors, which generalize to the present complex geometry.

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